INTERACTIVE TUTORIALS FOR UPPER LEVEL QUANTUM MECHANICS COURSES

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This thesis explores the ongoing need for interactive tutorials in the upper level undergraduate Quantum Mechanics course. It first summarizes the development and evaluation of tutorials at the introductory physics level by others, and then challenges the belief that upper level students do not need this type of intervention by citing research in student difficulties in learning Quantum Mechanics. Physics Education research shows that there are common student misconceptions that persist even in the upper level undergraduate courses such as Quantum Mechanics. Cognitive research serves as a guide for effective curriculum design. A description of the iterative process for developing and evaluating the tutorials is discussed. The development and evaluation of "The Time Evolution of a Wave Function" Quantum Interactive Learning Tutorial (QuILT) is described in detail. Finally, the success of the QuILT in reducing the common misconceptions about time evolution is discussed.

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PREFACE

Thank you to Dr. Singh for encouraging me to finish this degree. I began with great reluctance and the belief that writing a thesis about quantum mechanics could in no way benefit me professionally in my current teaching position. I was wrong, as the challenges of teaching are universal and independent of the clientele. Every year I put a quote up in my classroom that I refer to as my professional goal for the year – this year it will be “counter the passive learning environment of a typical course, and secure a mental commitment from the student.” (McDermott, 1991)

Thank you to my father who agreed to watch my three lovely children under the age of 5 while I wrote this thesis. Thank you to my mom, who coordinated everyone’s schedule in true sergeant fashion, and also helped with the kids. Thank you to my husband for all the laundry and dishes support, and for the late night talks about education issues. Thank you to Aunt Cindy for the technical support.

And finally thank you to my committee members and all my professors at Pitt who supported me throughout the years.

I dedicate this thesis to my children; may I show by example that what you start, you should finish.

1.0 INTRODUCTION

Our society increasingly relies on technology and our education system is feeling the pressure to produce a highly educated workforce. At the same time, education research shows that many of our traditional methods of teaching are ineffective. One of the main stumbling blocks to reforming our education system is the belief held by students, parents, instructors and administrators that the instructor should do most of the work for the student. Students at all levels of our education system often believe the instructor's primary role is to give information and answer questions in the form of lucid lectures. However, all the educational research states otherwise. The tutorials we are developing for undergraduate Quantum Mechanics courses change the instructor's role from a lecturer to a coach and the student's role from a passive learner to an active learner. In this thesis, I argue that to effect positive change in our educational system, we must employ research-based tools like the interactive Quantum tutorials.

Consider the following scenario: A professor lectures clearly and logically on new concepts, and works example problems which apply the concepts to a few situations. The students review lecture notes and complete homework. The professor writes an exam which requires the students to apply the concepts to new situations. The students come to the exam prepared and confident but leave wondering why the material on the evaluation seems so unfamiliar. The professor wonders why the students cannot think independently and apply the physics principles they learned in the lecture to new situations. This scenario is common throughout our educational system. Some students, who are like professors and have learned to interpret and apply concepts in new situations and organize their knowledge hierarchically on their own, prevail while others who need help in developing these skills struggle. Education research shows us that instruction can play a critical role in helping students develop higher order thinking skills and can help them organize and extend their knowledge structure.

The goals of the course, instructional design, and method of evaluation must be aligned with each other for learning to be meaningful. The desired outcomes can be broadly grouped into two areas which compete for limited time during a semester: *Science as a Body of Knowledge* vs. *Science as a Process*. If the priority is to recall definitions and reproduce proofs then instruction and evaluation should focus on those. Likewise, if the goal is to help students learn to think like a physicist and interpret and apply acquired knowledge flexibly in new situations, then instructional design and method of evaluation should reflect that. Unfortunately, traditional physics instruction in most introductory and advanced physics courses does not emphasize *Science as a Process* and mostly supports *Science as a Body of Knowledge.* The professor assumes that because the students have passively listened to the lectures, they will be able to make a great cognitive leap and figure out how to apply the concepts learned to new situations. Research has shown, however, that good teaching by an instructor in the form of a lecture alone is not enough for meaningful learning. Students need to be actively engaged in the learning process. They must be provided opportunities to practice the skills they must learn and they must be given feedback and scaffolding support as needed. Furthermore, this support must gradually decrease as the students develop self-reliance.

The goal of this thesis is to discuss the role of research based, interactive tutorials as an excellent tool for allowing students to move from a passive to an active role. The tutorials help students learn to reason systematically and build a robust knowledge structure. They help students view *Science as a Process* rather than *Science as a Body of Knowledge.* The lecture and tutorial activities can be combined as part of the instruction. During tutorials, students work in small groups and practice applying the concepts to new and unique situations *with support from their peers and instructor.* Common misconceptions and difficulties are explicitly brought out in the tutorials and students are provided guidance and support to organize their knowledge hierarchically where there is less room for misconceptions. After discussing the role of research-based tutorials as an effective learning tool at all levels of physics instruction, I discuss the development of a tutorial for teaching time-development of wave function in undergraduate Quantum Mechanics.

2.0 OVERVIEW OF EDUCATION RESEARCH

Educational Research shows that the traditional approach to teaching physics or "teaching by telling" does not work. In this section, we will cite educational research to assert that no matter how logical and engaging the lectures are in the traditional approach, exclusive use of this method will not allow students to gain meaningful knowledge. Novel teaching and learning methods such as interactive tutorials which combine the traditional and constructivist approach are needed to keep students actively engaged in the learning process and help them construct their own understandings. One major advantage of the tutorial approach is that it does not require a major change in the instructor's teaching style and hence can be adopted in a widespread manner. But before we discuss the tutorial approach in introductory physics and quantum mechanics, we will first review the traditional vs. constructivist approach and review some relevant prior research.

2.1 TRADITIONAL VERSUS CONSTRUCTIVIST APPROACH

In the Traditional teaching approach, lecture is the prevalent mode of instruction. First, the formalism is discussed by the instructor. A good instructor may then illustrate how the formalism applies using straightforward examples. Later, more complex examples may be used to illustrate how different concepts fit together. Due to years of experience and expertise teaching the topic, the instructor may even point out misconceptions and errors students have made in the past. The instructor may believe that this knowledge will prevent students from making similar mistakes. In the next class, the instructor may take students' questions on the homework assignments, and the class moves on to the next topic. This approach demands only deductive reasoning from the student; use of inductive and abstract thought processes, and making generalizations are not emphasized. Students tend to focus on finding an algorithm that works for problems of similar types, rather than focusing on the reasons for the steps in the algorithm (McDermott, 1990).

This traditional approach is acceptable to most physics majors and other students taking physics courses including non- majors (Tobias, 1990) because this is what they are used to. Moreover, the constructivist approach requires significantly more thinking on the students' part and humans try to minimize the cognitive load unless the topic is intrinsically interesting. Unfortunately, just like it is impossible to learn how to play the piano well by mostly listening to other people play, it is impossible for students to develop a solid grasp of physics concepts without having the opportunity to practice the skills with sufficient guidance and support from an instructor. Many successfully complete the course, but comment afterward that it wasn't interesting or relevant and they hardly remember anything. When students come out of their physics courses with the belief that physics is a collection of facts and formulas and is irrelevant for everyday life, the impact is felt through the whole educational system because many of these students will become future K-12 educators. Of course, the beauty of physics as a field of science is its ability to explain diverse phenomena in everyday experience in terms of a few basic principles. Unfortunately, traditional instruction does not provide adequate opportunity to students to organize their physics knowledge in such a way that they can see the coherence of physics and be able to apply the principles flexibly in different situations.

In the constructivist approach, the instructor starts by presenting a situation, and posing questions. The students observe, discuss, experiment, and ask for guidance. This bottom up instruction allows students to construct their own knowledge on the topic with guidance and support from the instructor based upon the students' prior knowledge. Formalism is introduced after exploration and generalization occur. This approach focuses on the students and keeps them active in the learning process. Many studies have shown that this approach is more effective for *all populations* of students (McDermott, 1990).

The tutorial approach is a combination of the traditional and constructivist approach. Research shows that exclusively traditional methods do not work but requiring students to discover all the material in an introductory course on their own in the constructivist fashion would turn a two semester course into a four year nightmare! The question remains: within the constraints of the current post secondary education system, how do we take the best of both methods and combine them into something that works for the students? One tool that has been developed by education research is the interactive tutorial. It is a middle ground between the traditional and constructivist approaches allowing for some top down instruction and some bottom up discovery.

2.2 A GROWING BODY OF RESEARCH INFORMATION

The interactive tutorial is an excellent teaching and learning tool, but it must be research based for it to work effectively. In the development of any educational material, research from different fields is employed. These fields can be broadly classified in three categories: 1. Cognitive Research (CR), which sheds light on how people learn and solve problems. 2.

Education Research (ER), which is conducted by researchers in the schools of education; it does not focus so much on the content but on the evaluation of the instructional strategies in general. The focus is often on the learning and teaching tool. 3. Physics Education Research (PER), which focuses (among other things) on the student's understanding of specific topics in depth and on devising and evaluating strategies to significantly reduce difficulties related to those topics. The following sections summarize the findings from the three fields that are relevant for the development and evaluation of the tutorials. Although presented here as three separate fields, they are intertwined; one cannot work in any one field without drawing upon the others.

2.2.1 Contributions from Cognitive Psychology

Relevant findings from cognitive research are labeled with different letters for ease in referring to them later when we discuss the process of developing an interactive tutorial.

A. Piaget's stages of cognitive development distinguish students who are at the concrete operational stage from those who are at the formal operational stage (Ginsberg and Opper, 1969). Students at the concrete operational level can conserve number, length, liquid, mass, weight, area, and volume when the object is present and can be manipulated. They can combine, separate, and transform the quantities. When students move to the formal level, they can use symbols and manipulate variables using abstract thought without the presence of concrete objects. At this level they become interested in the actual process of thinking. According to Piaget, students begin to operate at the formal level between 11 and 15 years. However, more recent research (Cole, 2000) shows that only 30 to 40 percent of students in their teens and early twenties can solve problems at the formal operational level. The age at which transition occurs between the two levels is a bell shaped curve with a large standard deviation (Siegler, 1996).

The fact that many students have not made a transition from the concrete to the formal level even in college has implications for the development of effective teaching and learning tools such as tutorials. Moreover, Piaget proposed that students should be provided "optimal mismatch" for cognitive growth with guidance and support for accommodation and assimilation of new concepts. The interactive tutorials strive to provide "optimal mismatch" based upon prior investigation of student difficulties.

B. While Piaget's contributions to cognitive science were focused on the individual, Vygotsky was a social educational psychologist and focused on the effect of social environment on learning. Vygotsky's notion of Zone of Proximal Development (ZPD) is defined as "the gap between what a student can achieve individually versus what s/he can achieve with the help of an instructor who is familiar with student's prior knowledge" (Vygotsky, 1978). When the gap between the prior knowledge of the student and the instruction is large, no meaningful learning occurs. However, if the instructor builds on the existing knowledge of students and takes small steps to stretch student's learning towards the desired goals of the course, the student can make great gains in knowledge. This process is often referred to as scaffolding, and is an integral component of the interactive tutorials.

C. Miller's groundbreaking work in the area of short term and long term memory has far reaching implications on development of instructional tools (Miller, 1956). Miller's research showed that only five to nine "bits" of information can be stored and processed in the short term memory (STM). For example, a nine digit phone number would be considered nine "bits" of information. Later research (Chase, 1973; Neves, 1981) shows that an expert in a field can store a much larger amount of information in a "bit" of STM by "chunking" different pieces of information together. A nine digit phone number can be chunked into an area code, a local exchange, and four digits. Nine bits of information are reduced to six. Similarly, physics experts develop deep connections between different concepts in mechanics such as displacement, velocity, acceleration, force, momentum, etc. Due to this "chunking" of information by a physics expert, a large body of knowledge can be recalled together and takes up just one "bit" in STM. For the novice physics learner, the connections are not yet made, and the six mechanics concepts listed above each require their own "bit" in STM. This implies that the student can have cognitive overload while going through a long chain of reasoning requiring more cognitive resources than is available if different instructional units do not build on each other and give students an opportunity to chunk knowledge gradually. Interactive tutorials use tools such as flow charts and concept maps to give students the opportunity to practice chunking information to reduce cognitive load and to promote robust knowledge structure.

D. Research on the effects of practice (Anderson, 1999) should make any teacher smile. It's what we've known intuitively all along – doing your homework helps! These results show that more practice yields faster and more accurate recall, and provides for better long term retention after the practice is stopped. Of course, the student must be practicing useful and appropriate skills – if the goal is to apply acceleration concepts in different contexts, reciting the definition of acceleration is not helpful. Just plugging numbers in equations is not helpful if the student does not take the time to think about the physics. Moreover, some approaches to practice prove to be more effective. Immediate recall was slightly better for a group that studied eight times in a day, compared to a group that studied twice a day for 4 days. However, long term retention was markedly higher for the latter group. Cramming the day before a test may give a slight edge on test day, but the recall drops quickly, and the information reviewed for the test is gone (Anderson, 1999). Gone with it is the opportunity to repair, extend and organize the LTM and make deep, rich connections. The spiral back method, where students apply a concept in several different contexts, and student elaboration, where students are given an opportunity to explain their reasoning, both prove to be effective ways of providing alternative pathways for recall (Anderson, 1999). Well designed interactive tutorials assure that the student is getting appropriate practice as opposed to mindless busywork.

E. We want our students to come to the table with ideas and observations about the world around them. But our work would be much easier if they came without some of their preconceptions. Whether a help or a hindrance, we must consider that "Students are not blank slates" (Shauble, 1990) and we must work to remove the wrong ideas and replace them with correct ones. It should be no surprise that students familiar with a world filled with friction think that there must be a constant force on an object to produce constant velocity. They have made a valid observation that in order to move an object across a table, they have to apply a force. Then they make an invalid conclusion that Newton's first law just does not apply in the presence of friction. The student's knowledge structure must be robust so that there is little room for misconceptions. Research has shown that simply showing students why the new ideas are applicable in a situation is not sufficient; students must learn why the old ideas do not apply (Posner, 1982). Interactive learning tutorials allow students opportunities to examine their misconceptions and then provide scaffolding support to build a robust knowledge structure.

One may wonder why students have a difficult time with physics courses in light of A through E above. In any given class, there may be students who may not have reached Piaget's formal level of thinking, students for whom the gap between what they know and the level at which instruction is targeted is insurmountable, students with cognitive overload because STM does not have resources available to process information, students who don't know how to study and practice effectively, and students with preconceptions about the world around them which hinder learning. The barriers to learning are large but surmountable, so good instructional tools such as research-based tutorials are needed to help students.

2.2.2 Contributions from Education Research and Physics Education Research

A major goal of Physics Education Research (PER) is to investigate the kinds of difficulties students have in learning physics and to develop and evaluate strategies to help students. Written tests are administered to a large population of students and in depth interviews are conducted with a subset of students. Information is gathered before, during, and after instruction, and can be used for two purposes: to assess the effectiveness of an instructional strategy and to guide the development or revision of a curriculum. After the intervention is revised, the process starts again. This continuous, interactive, iterative process has produced a wealth of information about how students learn physics effectively.

For the sake of this discussion, *meaningful learning* is defined as the "ability to interpret and apply knowledge in similar situations which are not identical to those in which it was initially acquired." After formal, traditional instruction, it is very common for students to have misconceptions about physics which are only detected when we check for meaningful learning. It has also been shown that the ability to solve a complex quantitative problem does not equate to a good conceptual understanding because students can often memorize algorithm without understanding (McDermott, 1992). In the physics field, *meaningful learning* occurs when the knowledge of physics concepts, the skills to represent the concepts, and the ability to reason using the concepts is developed simultaneously (McDermott, 1991). Research shows that the concepts should be developed in stages, and a spiral back approach going back to the old concepts is effective. Each topic should be applied to more than one situation, so that students don't compartmentalize. The practice of simply pointing out common misconceptions and reasoning errors is not effective: although the error may resolve itself in that context or for that problem, it simply resurfaces later in other contexts (Posner, 1982).

The process of inquiry has repeatedly been shown to be superior to instructional methods that present information as an inert body of knowledge. (McDermott, 1991) It lends itself to more peer interaction, which has many proven advantages: the articulation of an idea to a peer requires logic and organization. It forces the students to refine their thoughts. It allows students to remember concepts by recalling actual conversations. Students challenge the logic of arguments and work harder to justify their own logic when they interact with peers rather than teachers. They are more inclined to draw a picture or write an equation during their efforts (Rogoff, 1998). Research shows that peers can "co-construct" knowledge, meaning that two students working together could get a correct answer, even when neither one had the correct answer working independently (Singh, 2005). The interactive tutorials are designed with these pedagogical issues in mind.

3.0 THE PROCESS OF DEVELOPING AN INTERACTIVE TUTORIAL

In this chapter, the process of developing a tutorial will be discussed. The three part process consists of conducting investigations of student understanding, developing instructional strategies which directly address student difficulties and misconceptions, and iterating the process of designing, testing, modifying and revising the tutorials.

3.1 IDENTIFYING COMMON STUDENT DIFFICULTIES

Several strategies are used to identify and characterize the types of difficulties student encounter. Often at the beginning of a topic, individual students are interviewed to determine their understanding of a phenomenon. Discussions are audio taped for further analysis. Students are also monitored as they participate in labs and lecture discussions, and homework and exams are evaluated. Common difficulties are found for this group of students. Several professors and graduate students work together to formulate questions which explore the common difficulties. These questions are given to a larger group of students to determine if the common difficulties can be attributed to a local environment (such as professor’s teaching style, textbook, or university) or if they represent a more pervasive problem. These common difficulties are the basis for tutorial development.

Research in introductory physics has shown that common difficulties exist for students enrolled in calculus based, algebra based, and non-science major courses. These common difficulties persist after formal traditional instruction for all three groups, as they do with students enrolled in overseas universities (McDermott, 1992). All of this implies that the difficulties are universal. The skill sets and levels of preparation that students bring to the introductory courses vary widely, and we cannot assume that the students will overcome these obstacles on their own. If we, as educators, want students to acquire *meaningful learning* in the field of physics, we must make our teaching student centered.

3.2 POSSIBLE REASONS FOR COMMON STUDENT DIFFICULTIES

Once common difficulties related to a topic have been identified, a tutorial is developed in an attempt to help students build robust knowledge structure where there is less room for such difficulties. The tutorial must also address specific misconceptions, and take in to account the issues discussed in section 2.2.

Common difficulties related to a particular topic arise for several reasons. When a robust knowledge structure related to a topic is lacking, students try to over generalize their knowledge resources that are appropriate in some situations to cases where they are not applicable. Cognitive research shows that the way people over generalize their knowledge resources in a particular context based upon their experiences is very similar and leads to common misconceptions and difficulties. Moreover, students who are learning a topic for the first time may lack a conceptual model especially because they may not have hands on experience with the phenomenon. Students may also lack experience or practice reasoning about the concept. In order to overcome the difficulties, they should be given opportunities to experience the phenomenon if possible, and be put in a situation where they can practice reasoning about it. It is the job of the curriculum developer to create these situations, and the job of the instructor to facilitate them. It is NOT the job of the instructor to answer every question students ask in great detail. As a matter of fact, doing so can interfere with ability of students to construct their own knowledge. Providing appropriate scaffolding and guiding students with appropriate hints has been found to be more effective in helping students build a robust knowledge structure.

An interactive tutorial **based on research** should do the following things as appropriate:

- Start with a misconception which was identified through research.
- Provide multiple contexts for the student to experience the phenomenon and practice reasoning about the phenomenon.
- Provide scaffolding which helps students start with their prior knowledge and gives them opportunity to build on their prior knowledge.
- Promote chunking of concepts by exploring the interconnections between similar ideas.
- Provide opportunities to replace preconceptions with a robust knowledge structure so that students understand why the preconceptions are wrong.
- Provide opportunities to interact with peers and co-construct knowledge.

The QuILT shown in the appendix, titled "The Time Evolution of a Wave Function" strives to accomplish these goals. A description of its development and an analysis of its effectiveness follow in chapter 5.

More than any other field of physics, Quantum Mechanics fits the description of "no direct hands on experience" with microscopic world and "no experience or practice reasoning about it" outside of the class. It is a field that would benefit greatly from more interactive learning tools. Chapter 4 discusses this need.

4.0 RATIONALE FOR INTERACTIVE TUTORIALS AT THE ADVANCED UNDERGRADUATE LEVEL

Quantum Mechanics is abstract. Like Physical Chemistry for chemistry majors, and Organic Chemistry for nursing majors, it is commonly referred to as the "weeding out" class. It is the course that causes many students to get off the PhD track, and head for degrees in related fields. Richard Feynman himself said "I think I can safely say that nobody understands quantum mechanics" (Feynman, 1945).

Commonly used textbooks include warnings for students of the difficulties associated with the course (Griffiths, 1995). "There is no general consensus as to what its fundamental principals are, how it should be taught, or what it really *means.*" Griffiths goes on to say, "Not only is quantum theory conceptually rich, it is also technically difficult, and exact solutions … are few and far between." Midway through his preface, he invites students who want to know what quantum mechanics *means* instead of just what it *does*, to read chapter one and then skip to the afterward to get an idea of how mysterious it is. Now, like any good student, I made a mental note to do just that. But by the time I was introduced to the Schrödinger Equation, the three different statistical interpretations, and normalization of a wave function, all my bits of STM were full, and my good intention of reading the afterward to learn what quantum mechanics really *means* was forgotten. Besides, I had a long homework assignment in each of my classes and didn't have time for extracurricular pursuits – I had to get good at solving the Schrödinger equation for different potentials and grade my papers for my TA assignment. There was no time to think about what it really means – a pitfall into which many of our students fall.

In my textbook (Griffiths), the following quote is highlighted with a frowny face next to it, and the word "ugh" written in the margin: "…quantum mechanics is not, in my view, something that flows smoothly and naturally from earlier theories. On the contrary it represents an abrupt and revolutionary departure from classical ideas, calling forth a wholly new and radically counterintuitive way of thinking about the world. That, indeed, is what makes it such a fascinating subject." The counterintuitive nature of the course also makes it the perfect upper-level subject for the development of interactive learning tools to help students.

4.1 THE EXPERT'S VIEW OF QUANTUM MECHANICS

A professional physicist may make the case that doing Quantum Mechanics is actually easier than some other topics in upper-level undergraduate physics. Compared to Electricity and Magnetism (E&M), quantum mechanics may be easier because it deals with one field, and that field is a scalar field rather than a vector field. In Classical Mechanics, one solves Newton's Second Law for a given potential, applies the boundary conditions, and writes the solution. Quantum mechanics differs in the fact that one solves the Schrödinger Equation rather than Newton's Second Law. Many professional physicists feel that, with a solid background in differential equations, linear algebra, and probability and statistics, doing Quantum Mechanics can become routine. But there is a disparity between how difficult students perceive the material and how difficult the instructors believe it is to learn. Again, this makes it an excellent subject area for developing tutorials which will bring the instructor and the student together in Vygotsky's Zone of Proximal Development.

4.2 SOURCES OF COMMON DIFFICULTIES IN QUANTUM MECHANICS

As mentioned in section 3.2, there are several reasons for common difficulties related to a particular topic. The lack of a conceptual model due to a lack of hands on experience with the phenomenon and a lack of experience or practice reasoning about the concepts in everyday experience makes learning quantum mechanics even more challenging. In general, students starting an introductory course have seen rainbows, used electricity, cooked on a stove, driven a car, and played contact sports. These everyday experiences give them a context for learning electricity, heat, and kinematics. There are no corresponding everyday experiences to familiarize students with quantum phenomenon. For the introductory course, additional hands on experience can be easily created in the instructional setting with demonstrations. For the quantum course, computer simulations are the main resources to help students visualize quantum mechanical concepts.

For example, consider the question of the minimum safe speed for a train at the top of a roller coaster loop. When you ask a group of students what they think this speed will depend upon, the mass of the object is always one of the responses. After the traditional treatment of applying Newton's Laws and showing that mass cancels on both sides of the equation, many students will be convinced that mass does not matter in calculating the velocity and they can use the principles successfully to solve quantitative problems. For the students who don't quite believe the algebra, and still feel intuitively that mass should matter, the instructor has the very effective option of asking the student "Have you been on a coaster with someone heavier or lighter than you? Did the mass matter?" The instructor can guide students and focus their attention back on the equation, and students may make the connection. The opportunity to practice reasoning about the concept, and to draw on their experiences is a very powerful teaching tool (Laws, 1997). In Quantum Mechanics, the concrete student experience, analogous to the roller coaster ride for introductory physics students, does not exist. The running joke with my classmates when I was taking quantum mechanics was, "What *is* the particle in the box? Can we have an in-class demonstration to visualize it?" An interactive research-based tutorial is not a demonstration but gives students an opportunity to build on what they know in a systematic manner and addresses the common misconceptions explicitly.

5.0 APPLYING RESEARCH FINDINGS TO QUANTUM MECHANICS INSTRUCTION

Section 2.2.1 summarized major findings of cognitive research. In this section I will discuss how four of these five findings impact the learning of Quantum Mechanics.

A. As mentioned in section 2.2.1.A, as few as 30 to 40 % of thirteen to twenty-four year olds are able to solve problems at the formal operational level. This is a major roadblock to teaching the introductory course. My years of experience teaching from middle school to undergraduate level lead me to believe that educators waste a great deal of time and money attempting to teach abstract concepts to students who are still at the concrete level. One would think that any student who has successfully made it through an intermediate level undergraduate physics curriculum to the upper level course such as quantum mechanics *must* be operating at the abstract level. While most students in the quantum mechanics course can do more abstract reasoning than a typical introductory physics student, the level of abstractness in a quantum mechanics course increases significantly compared to introductory courses and presents a challenge for many students.

B. Finding of cognitive research in section 2.2.1.B discussed the notion of the Zone of Proximal Development. One of the most difficult challenges for an instructor is to determine what the students know at the beginning of the course, and build on that knowledge. For the introductory course, there are two possibilities: either students had a high school physics course, or they didn't. Certainly the differences in the skills that the students bring to the course are large in introductory courses, but research shows that spread in students' preparedness for advanced courses remains large as the students move through the curriculum. Remembering my first Quantum course of about 15 students, half were undergraduates taking the course for the first time. Of the graduate students, about half were foreign students who had already earned a masters degree in their home county – i.e. they had already completed a comparable course. Some had already taken Modern Physics, some were taking it concurrently, and two had never had Modern Physics. The mathematical background of the students varied from no Mathematical Methods course to a few students who had completed two semesters. Six of the students were taking linear algebra during the same semester, and one was taking a statistics course. Any professor who faces such a diverse level of student preparation can particularly benefit from tools such as tutorials that help students start with what they know, and build on that knowledge.

C. Even the upper-level students in quantum mechanics courses can have issues with cognitive overload. There must be some cognitive discomfort during the learning process just as there must be some muscular discomfort during a weight lifting workout. However, if it hurts too much, you are impeding your progress. Similarly, cognitive overload can impede learning.

In an abstract course such a quantum mechanics, the organizing of information into chunks and storing in LTM may be more challenging than introductory physics. As an example, compare the concept of "velocity" in introductory physics and the concept of "expectation value of momentum" in quantum mechanics. For introductory students, the concept of velocity may require different bits of STM for displacement, time, ratio, and appropriate units. Once this is sorted out in STM, connections can be made to acceleration and displacement. All of these concepts account for only six "bits" of STM, and students may still have the capacity to reason about these topics.

Velocity is a fundamental idea at the introductory level. Advanced students in quantum mechanics may come to the class believing that velocity (and eventually momentum) should be a fundamental idea at the quantum level and it should be an attribute independent of the position of the particle. However, what they discover is that in quantum mechanics position and momentum (or velocity) are operators rather than being deterministic variables as in classical physics. Students learn to find the expectation value of momentum in a given state since the momentum of a particle in a given state is not well-defined and is probabilistic. To truly follow the treatment of this concept of expectation value of momentum, students must have room in STM for momentum, wave function, probability density, expectation value, integration by parts, measurement and collapse of wave function, and average of measurements of momentum on large number of identically prepared systems. Not only do these concepts take up several "bits" of STM, these concepts are much more complicated than the concept of velocity in classical physics. Even for advanced students it is difficult to chunk these ideas together without guidance. To add to the confusion, at the end of the treatment, students learn that the expectation value of velocity (or momentum) is not the velocity (or momentum) of the particle. Griffith's notes in his quantum textbook: "Nothing we have seen so far would enable us to calculate the velocity of a particle – it's not even clear what velocity *means* in quantum mechanics" (Griffith, 1995). When asked what expectation value of momentum means, most students can reply, "it's what you get when you sandwich the momentum operator between Ψ^* and Ψ, and integrate over all space." Their ability to recite this fact, however, does not mean they fully understand expectation value. (Singh, 2008) For example, many students had difficulty understanding that the expectation value of momentum is the average of large number of measurements performed on identically prepared system.

D. Effective practice is critical for meaningful learning and is the corner stone of a good course. Often during the process of developing tutorials, it occurred to me that perhaps all that needs to be done is to slightly modify the exercises that already exist in popular textbooks. In an effort to evaluate the extent to which exercises given to beginning Quantum Mechanics students promote *effective learning,* I randomly picked problem 1.7 on pg 13 of Griffith's text to analyze in terms of Anderson's cognitive research findings. Here is the problem:

At t = 0, a particle is represented by the wave function

$$\Psi = \frac{Ax}{a}, if\, 0 \le x \le a$$
$$\Psi = \frac{A(b-x)}{(b-a)}, if a \le x \le b$$
$$\Psi = 0, otherwise$$

Where A, a, and b are constants.

a. *Normalize Ψ, that is, find A in terms of a and b.*
b. *Sketch Ψ(x,0) as a function of x.*
c. *Where is the particle most likely to be found, at t=0?*
d. *What is the probability of finding the particle to the left of a? Check your result in the limiting case of b = a and b = 2a.*
e. *What is the expectation value of x?*

I set out to analyze the problem in terms of Anderson's criteria for effective practice. Long, quantitative exercises do not promote meaningful learning if one does not reflect upon the problem solving process. Qualitatively rich problems or quantitative problems coupled with qualitative or conceptual problems are more likely to promote learning. For meaningful learning, the need to develop mathematical skills must be balanced with the need to develop conceptual understanding. On this problem, I spent 2.5 hours working independently, with a peer, and independently again. I took advantage of my peer's strengths, and I checked to make sure my answer made sense to me in parts a, c, and e even though the problem did not instruct me to do so.

Because I took these actions, the exercise became both mathematically rigorous and qualitatively rich. However, my mindset while working this problem was most likely different from typical students actually enrolled in a course. Eighteen years of teaching has shown me that even the most conscientious student often does not take the extra step to make an existing quantitative exercise qualitatively rich. Students get the assignment done so that it can be turned in on time. This exercise has the potential to promote meaningful learning if one takes the time to reflect upon why certain physics principles were employed for the problem and how one can recognize in the future that another problem with different context should be solved using similar method However, most students are likely to miss the opportunity. The qualitatively interesting part of the problem chosen is the comparison between the answer for parts C and E. One common difficulty found at the quantum level is a student's inability to distinguish between individual measurement and expectation value. (Singh, 2008) This problem provides the framework to explore this further. Adding a "part F" that asks the student to explain how C and E are different may be a good start.

E. Ironically, well prepared students who have had extensive practice reasoning about introductory concepts may be at a disadvantage learning quantum mechanics! The great difficulty of removing student misconceptions discussed earlier is complicated by the fact that some ideas which are correct for macroscopic systems are incorrect for microscopic systems. It is not just a matter of removing a misconception; students have to be convinced that the idea is correct in one situation, incorrect in another, and must be able to distinguish between the two! An example is the phenomenon of tunneling at a potential barrier. In the macroscopic world, when an object approaches a potential energy barrier with $E < V$, the probability of detecting the object on the other side of the barrier is zero. In the microscopic world, there are non-zero transmission and reflection probabilities, and the phenomenon of tunneling is exploited in modern technology.

Students need multiple opportunities to practice these non-intuitive concepts in different situations. A clever application of this idea (Griffith, 1995) uses an infinite well rather than a potential barrier, and asks students to consider the probability that Quantum Mechanics will allow you and your car to bounce back from the edge of a cliff. Again, this exercise provides great material for an interactive tutorial.

6.0 THE TIME EVOLUTION OF A WAVE FUNCTION QUANTUM INTERACTIVE LEARNING TUTORIAL (QUILT)

6.1 IDENTIFYING COMMON DIFFICULTIES

The first step of creating this interactive tutorial was researching the common student difficulties on the topic of time development. This research was done during the late 90's by Dr. Singh and her group at the University of Pittsburgh (Singh, 2001). The research included 89 students at 6 universities. The test covered quantum measurement, time development of wave functions, time dependence of expectation values, the statistical nature of quantum mechanics, and the Copenhagen interpretation. The questions were adapted from existing exams and homework. A preliminary version was given to a group of students at Duquesne University, and the test was modified after extensive class discussion and individual discussion with volunteers. This modified, 50 minute test is included in Appendix B.

An in-depth analysis of students' misconceptions was performed by interviewing nine students as they worked through the test using a think aloud protocol. In this protocol, students talked aloud while answering the questions without much interruptions and the interviewer asked them for clarification of the points they had not made clear only at the end. The common difficulties found stem from a tendency to over generalize and an inability to discriminate between related concepts. The common difficulties found in the interviews were the same as those found by administering the written test to eighty-nine other students. They are summarized as follows:

- Students have difficulty distinguishing between individual measurements of a physical quantity and its expectation value.
- Students often believe that a system in any eigenstate (even if it is not an energy eigenstate) does not evolve in time.
- Students believe that if the expectation value is zero in the initial state, then the expectation value cannot have any time dependence. This misconception is similar to the misconception in introductory physics that if the velocity of a particle is zero in an initial state then the rate of change of velocity is zero as well.
- Students are often confused about the role of eigenstates of the Hamiltonian operator (energy eigenstates) in the time evolution of the system and the eigenstates of other operators.
- Students believe that since the time evolution of a wave function is of the form $e^{\frac{-i\hat{H}t}{\hbar}}$, it cannot change the probability of obtaining an outcome when an observable is measured.

We can broadly classify knowledge deficiencies in three categories: lack of knowledge, knowledge that can be retrieved but not interpreted correctly, and knowledge that can be retrieved and interpreted at the basic level, but cannot be used to draw inferences. The difficulties above represent knowledge deficiencies at all three levels. Often instructors do not provide the scaffolding needed to build meaningful knowledge structure and students can end up with knowledge deficiency. If the curriculum is designed to address the three levels of knowledge deficiency, the students may develop robust understanding of quantum mechanics concepts.

Most of the introductory physics misconceptions are actually *preconceptions* which are often formed by the student's interpretation of everyday phenomenon. These preconceptions are difficult to correct and reappear in different contexts if they are not addressed appropriately in the curriculum. In Quantum Mechanics, there are few preconceptions – students simply don't have everyday experiences with quantum phenomenon. However, the misconceptions in quantum mechanics could be due to the difficulty in reconciling the quantum mechanical concepts with the classical concepts learned in classical physics courses or from everyday experiences. Students may also have difficulty in distinguishing between related concepts and over generalize the knowledge gained from a Modern Physics course or learned previously in the same course. For example, students often over generalize and believe that position eigenstates (in which the position of the particle has a definite value) are the same as energy eigenstates (in which the energy of the particle has a definite value). Similarly, students over generalize Bohr model and often believe that even in the full quantum mechanical model, both the energy and position of an electron in a Hydrogen atom can be well-defined simultaneously.

6.2 INITIAL DESIGN OF THE TIME EVOLUTION OF A WAVE FUNCTION QUILT

With these common difficulties in mind, and the findings of Cognitive Research and Physics Education Research as a guide, we began the design of the **Time Evolution of a Wave Function** QuILT. Many different versions were written, and iterations occurred after feedback from small groups of students, and after interviews with students using the think aloud protocol and working on the tutorials.

6.2.1 The First Version

The first version gives an initial wave function $\Psi(x,0) = Ae^{-ax^2}$, and asks the students to write the energy eigenstates, write the initial wave function as a linear combination of the energy eigenstates, describe the process of finding the expansion coefficients C_n's, and write an expression that would have to be evaluated to find a $\Psi(x,t)$. The question is repeated three times in three different contexts: an infinite square well or ISW (of course for this potential energy, the initial state is not possible since the wave function must go to zero at the boundaries of the well but a Gaussian initial state only decays to zero at positive and negative infinity), a harmonic oscillator, and a free particle. These questions were designed to point out the importance of the initial state on the time evolution of wave function. When $\Psi(x,0)$ is an energy eigenstate of the system, the time evolution is easy to express. When $\Psi(x,0)$ is not an energy eigenstate of the given system, the time development of wave function involves a linear superposition of energy eigenstates. After discussion, we determined that the confusion about the role of energy eigenstates needed to be addressed very early in the semester, and could not wait until students had learned about all the three different potentials above. The set of questions in the preliminary tutorial might make the basis for a good midterm exam, but was not suited for an early semester interactive tutorial.

The second section asked the student to consider an infinite square well with two different initial wave functions: $\Psi(x,0) = A\sin(kx)$, $\Psi(x,0) = A\Psi_1 + B\Psi_2$, where $k = \frac{n\pi}{a}$, a is the width of the potential well, and Ψ_1 and Ψ_2 are the first and second energy eigenstates of the ISW. For each initial wave function, students were asked to write down the energy eigenstates and energy eigenvalues, write the initial wave function as a linear combination of the energy eigenstates, describe the process of finding the C_n's, and write an expression for $\Psi(x,t)$. The solutions for this potential were worked out in the lecture, and the low mathematical difficulty of ISW allows for more cognitive effort to be directed to resolving the common difficulties surrounding energy eigenstates and time evolution. These questions eventually became the framework for the final version of the tutorial. They give students an opportunity to overcome the knowledge deficiencies related to lack of knowledge and lack of ability to apply the knowledge properly. They set the stage for questions that provide practice in applying knowledge in different situations and making inferences.

The third section used the time-dependence of expectation value $\frac{d<Q>}{dt} = \frac{i}{h} < [H,Q] > + < \frac{dQ}{dt} >$ to explore whether or not the expectation value of an observable is time dependent for a given potential and initial wave function. This section was revised quite a bit in the second version.

6.2.2 The Second Version

The second version began with a summary of time evolution of wave function as shown below: *The time evolution of a system can be found for any initial wave function even if it is not an energy eigenstate (or stationary state or solution of Time Independent Schrödinger Equation) by using the following steps:*

First: *Write the initial wave function as a linear combination of the energy eigenstates:* $\Psi(x,0)=\sum C_n\Psi_n$

Second: *Solve for the expansion coefficients* C_n *by multiplying both sides by* Ψ_m^**, and integrating over all space.*

Third: *Tack on the time dependence to each energy eigenstate as* $e^{\frac{-iE_nt}{h}}$.

Beginning this version with a summary of time evolution does several things for the student. It reduces the cognitive load and frees more STM to reason about the concepts. This summary was expanded, and eventually became a flow chart in the latest version.

Unfortunately, students are often instructed to use an algorithm to solve a problem, and do not understand why the steps work. A section of the tutorial gives them the opportunity to clarify the reasoning. The students start with the Time Dependent Schrodinger Equation (TDSE), and are instructed to "put the equation in a slightly different form, by gathering all the time dependence on the right, and all the Ψ dependence on the left, and integrating both sides." This allows them to see the integral form of the TDSE, and asks them to describe in words how the Hamiltonian operator relates to the evolution of the system in time. Next, the students are asked to consider two situations; the initial wave function is written as a linear combination of momentum eigenstates, and the initial wave function is written as a linear combination of energy eigenstates. The students are told that both expressions are correct, but one is more useful for finding the time-dependence of wave function than the other. They are asked to explain why. The students are guided and learn that the correct expression is written for the wave function at some later time easily when the wave function is written as a linear superposition of energy eigenstates (stationary states): $\Psi(x,t) = \sum C_n e^{\frac{-iHt}{\hbar}} \Psi_n$. This type of open ended question brings out many misconceptions. Although this series of questions was not used verbatim in the latest version, the student responses gathered during this part of the development of the QuILT were useful in forming the multiple choice questions used in the present version.

The third section starts by stating that the time dependence of expectation value is $\frac{d<Q>}{dt} = \frac{i}{\hbar} < [H,Q] > + < \frac{dQ}{dt} >$ and begins with a summary in the same style as the first section. The following two important points are inferred from the time-dependence of expectation value:

<u>First:</u> *If [H,Q] = 0, then d<Q>/dt= 0. This implies that expectation value <Q> does not evolve with time regardless of the initial state if the Hamiltonian commutes with Q. In this case, Q is a constant of motion.*

<u>Second:</u> *If [H,Q]≠0, there is <u>still</u> a situation in which d<Q>/dt= 0, and therefore <Q> does not evolve with time. This occurs when the particle is in an energy eigenstate of the system.*

Similar to section one, the summary is designed to reduce the cognitive load and leave resources available in STM to practice reasoning for different physical observable Q for different initial states for a given potential energy. The students learn that the initial state matters. A series of potentials, initial wave functions, and observables are given. They are summarized in the table below. $A\Psi_1 + B\Psi_2$ refers to the first and second energy eigenstate for the ISW, and a is the width of the potential well. HO stands for harmonic oscillator potential.

Table 1. List of activities in the second version of the QuILT

Observable	Potential	Initial state
Momentum	ISW	$\Psi(x,0) = A\sin(\frac{3\pi x}{a})$
Momentum	ISW	$\Psi(x,0) = A\Psi_1 + B\Psi_2$
Momentum	HO	$\Psi(x,0) = A\sin(\frac{3\pi x}{a})$
Momentum	HO	$\Psi(x,0) = Ae^{-ax^2}$
Position	ISW	$\Psi(x,0) = A\sin(\frac{3\pi x}{a})$
Position	ISW	$\Psi(x,0) = A\Psi_1 + B\Psi_2$
Position	HO	$\Psi(x,0) = Ae^{-ax^2}$

The students are led through the process of us ing the expression to calculate the tim e-dependence of expectation value for each of the observables for the given potential energy and given initial state of the system and make sense of what they obtain. The following questions are asked for each case:

- What is the Hamiltonian for the system?
- Does the Ha miltonian comm ute with the operator corresponding to the observable?

- What is the energy eigenstate for the given potential?
- Is the initial wave function an energy eigenstate for the given potential?
- Is the expectation value of the observable time dependent or time independent?

This process of modeling the thinking process may be called "hand holding" by some educators who feel that students should be able to develop these skills on their own. But a guided approach which provides appropriate scaffolding is necessary to help students build robust knowledge structure. There is great debate in Education Research about how much leading and modeling students need to become proficient with these types of reasoning skills. Preliminary research (Singh, 2001) shows that traditional lecture on this material does not allow students to become proficient in reasoning about the time-dependence of expectation value.

6.2.3 The Third Version

In the previous versions, when students were instructed to "put the equation in a slightly different form, by gathering all the time dependence on the right, and all the Ψ dependence on the left, and integrating both sides," a very common response was $\Psi(x,t) = \Psi_0 e^{\frac{-iHt}{\hbar}}$. This prompted us to write three more questions:

- *If you wrote your answer in the form* $\Psi(x,t) = \Psi_0 e^{\frac{-iHt}{\hbar}}$ *, you have done something wrong. What? (Hint: what is H? Why must it be written to the left of* Ψ_0 *?)*
- *Why is the Hamiltonian H called the time evolution operator?*
- *If you know the state of a system at t = 0, the state at time t is given by* $\Psi(x,t) = e^{\frac{-iHt}{\hbar}}\Psi_0$. *Can you think of a state* Ψ_0 *for which the operator*

$e^{\frac{-i\hat{H}t}{\hbar}}$ *acting on* Ψ_0 *will give a number times the same state? (Hint: Only for very special states will this operator acting of the state give a number. If the initial state is not the special state, we will not get a number times the same state. We will need to write the initial state as a linear superposition of those special states first if we do not want to write the wave function after time t in terms of the Hamiltonian operator.)*

This is an example of directly addressing a misconception. Even a student who wrote the expression correctly the first time would benefit from going through this thought process.

6.3 THE CURRENT VERSION

6.3.1 Rationale for Changes

Education Research has shown that students' ability to reason about a concept develops faster when they are given opportunity to think about different contexts in which the concept is applied and also given opportunity to think about the concept abstractly. Consider Part B, questions 1 through 4 of the current version in the Appendix. The responses to choose from do not specify a particular potential or a particular observable. The lack of context is by design. Part A gave a context for each problem, part B represents an attempt to gradually decrease the guidance and let students generalize the concept.

The Roman numeral choices for these four questions came from written student responses to earlier versions of the tutorial, and from interviews of students working through the tutorial using the think aloud protocol. The decision to move from open ended to multiple choice was made to take advantage of these common misconceptions. Students benefit from discussing why statements are incorrect just as much as discussing why statements are correct. It also allows for more objectivity in grading responses. We will eventually convert the tutorial to an online tool. In the multiple choice format, when students click a response, a help window will pop up. The text in the help window will address the reasoning errors and misconceptions that lead to the incorrect response.

Very few topics in Quantum Mechanics lend themselves to hands on experiences. The addition of a computer simulation in this tutorial is the next best thing. The tutorial directs the student to consider a wave function in an initial state and a specific potential. Students then make a prediction about the wave function's time evolution. The computer simulates the situation, and the students get immediate visual feedback on their prediction. The tutorial then guides students in reevaluating their thinking. For example, students often predict that the probability density after a time t for both stationary and non-stationary state wave functions will be time-independent because they forget that they must first write the non-stationary states as a linear superposition of stationary states before tacking on the time-dependence to each term in the expansion. Thus, the time-dependent cross terms will survive when probability density is found after time t for a non-stationary state wave function. When students look at the computer simulations for the time-dependence of probability density for two different initial wave functions, one of which is a stationary state wave function and the other is a non-stationary state wave function, they are in a state of disequilibrium. They observe that there is no time dependence to the probability density for the stationary state but there is time-dependence to the probability density for the non-stationary state. At this point, the tutorial provides guidance and helps students understand that since the Hamiltonian operator governs the time evolution of the system, eigenstates of the Hamiltonian (stationary states or energy eigenstates) are special when it comes to time-development issues. Then students learn the procedure for how to find the wave function after a time t for many different initial non-stationary state wave functions that correspond to different potential energies.

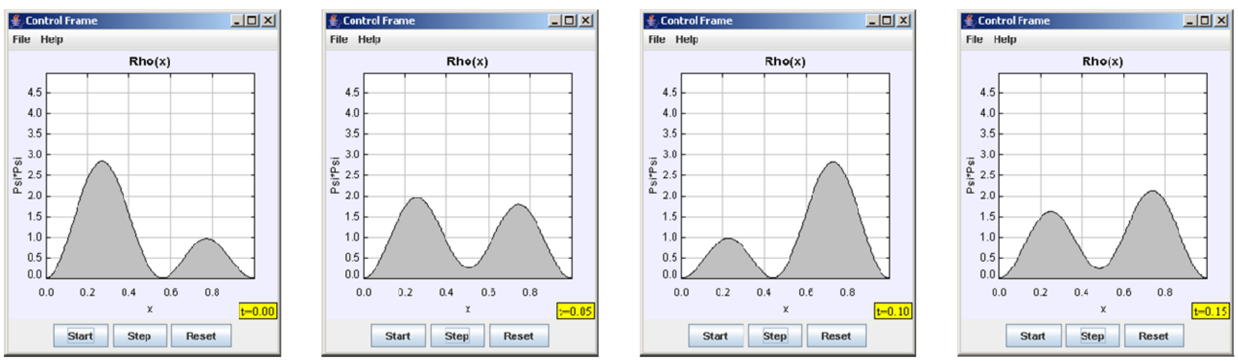


Figure 1: Computer simulation of time evolution of a wave function.

In the current version, the only context used for the pretest is the one dimensional infinite square well (ISW). This decision was made so that the QuILT could address difficulties with time evolution earlier in the semester. These difficulties are common and persistent even at the graduate level (Singh, 2008). In most sequences used in the introductory quantum course, the stationary states are introduced, and then several potentials are introduced in turn – the first usually being the ISW. Earlier intervention seemed to help students solidify the concept of time evolution, which in turn benefited the students for the rest of the semester.

Another major change is the addition of a flow chart for computing the time evolution of a state. This chart is at the end of Part B of the tutorial in the appendix, and summarizes the concepts used in Part A and B. It gives the formalism for the discrete and continuous states side by side in column format. Physics Education Research shows that when students construct their own concept maps and flow charts, they gain robust knowledge and relate ideas to allow for better chunking of material. To avoid cognitive overload, the decision was made in this case to create the flow chart for the students. The students can still benefit from using the chart.

6.4 IMPLEMENTATION AT THE UNIVERSITY OF PITTSBURGH

The final version shown in Appendix A was administered to 9 students in the undergraduate Quantum Mechanics course at the University of Pittsburgh. After traditional instruction, part A was given as a pretest. Students then worked in small groups to complete Part B. If students could not finish during the class period, they finished part B as homework. At the next class meeting, Part C was given as a posttest, and counted as a quiz grade for the class. The average pre to posttest scores improved from 53% to 85%. One of the most interesting results is from the two students who were absent the day of the pretest and tutorial – they scored 0% and 30% on the post test (Singh, 2008).

Along with the time development QuILT, 12 students also worked through a tutorial on the uncertainty principle and the Mach-Zehnder interferometer. The pretest to posttest improvements were 42% to 83% and 48% to 83 %, respectively. It seems that the QuILTS help students learn. Moreover, the students like them ! A survey was given to the class to determine how effective students found the time they spent on the tutorial. The responses were almost unanimously positive. Below are the questions and responses (Singh, 2008):

1. *Please rate the tutorials for their overall effectiveness where 1 means totally ineffective and 5 means very effective.* Average response 4.42
2. *How often did you complete the tutorial at home that you could not complete during the class? (1) never, (2) less than half the time, (3) often, (4) most of the time, or (5) always.* Average response 3.91
3. *How often were the hints provided for the tutorial useful?* Average response 4.52.
4. *Is it more helpful to do the tutorials in class or would you prefer to do them as homework? Please explain the advantages and disadvantages as you see it.* Ten students responded that doing the tutorials in class was more useful because the group discussion allowed them to focus on conceptual understanding. They saw the advantage of working with their peers and instructor. The two who preferred to work on them independently felt that they would put more time and effort into them at home.
5. *How frequently should the tutorials be administered in the class? Explain your reasoning.* All the students preferred once a week. Two reasons were given. The first was that the concepts learned in the tutorial helped them understand the homework problems. The second was that the tutorials helped them focus on concepts that were missed in other contexts, like lecture, homework or office hour discussions with the instructor.
6. *Do you prefer a multiple choice or open ended format for the tutorials? Explain your reasoning.* This was a question that elicited a varied response, but most

students saw the benefit of both types of questions even if they preferred one over the other. The multiple choice questions allowed students to focus on common difficulties and fundamental concepts, and they preferred multiple choice for the warm up. The open ended questions stimulated creative thought, allowed students to apply concepts and deepen understanding. Students preferred open ended for the main tutorial.

6.5 IMPLICATIONS FOR EDUCATION REFORM

Most educators actively working in the classroom, from K-12 to upper level graduate instructors, do not have the time to develop effective teaching tools. The way to reform our education system is to develop and evaluate interactive tools that can be easily implemented in the classroom without requiring a major change in the instructor's lecturing style. We have developed research-based tutorials in quantum mechanics to help students develop a solid grasp of quantum mechanics. The tutorials can be used in conjunction with regular lectures. Preliminary evaluations of the time-development of wave function QuILT shows that the tutorial is effective in helping students in developing a functional understanding of quantum mechanics concepts related to the dynamics of wave function.

APPENDIX A

The Time Evolution of a Wave Function

• A "system" refers to an electron in a potential energy well, e.g., an electron in a one-dimensional infinite square well. The system is specified by a given Hamiltonian.

• Assume all systems are isolated.

• Assume all systems have a time-independent Hamiltonian operator $\hat{H}$.

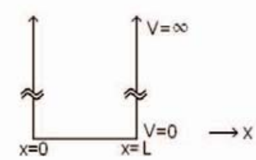
• TISE and TDSE are abbreviations for the time-independent Schrödinger equation and the time-dependent Schrödinger equation, respectively.

• The symbol $\sum$ in all questions denotes a sum over a complete set of states.

PART A

• Information for questions (I)-(VI)

In the following questions, an electron is in a one-dimensional infinite square well of width L. (The stationary states are $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$, and the allowed energies are $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, where $n = 1, 2, 3...$)



(I) Suppose the wave function for an electron at time $t = 0$ is given by $\psi(x,0) = \sqrt{2/L}\sin(5\pi x/L)$. Which one of the following is the wave function at time t?

(a) $\psi(x,t) = \sqrt{\frac{2}{L}}\sin(5\pi x/L)\cos(E_5 t/\hbar)$

(b) $\psi(x,t) = \sqrt{\frac{2}{L}}\sin(5\pi x/L)\,e^{-iE_5 t/\hbar}$

(c) Both (a) and (b) above are appropriate ways to write the wave function.

(d) None of the above.

(II) The wave function for an electron at time $t = 0$ is given by $\psi(x,0) = \sqrt{\frac{2}{L}}\sin(5\pi x/L)$. Which one of the following is true about the probability density, $|\psi(x,t)|^2$, after time t?

(a) $|\psi(x,t)|^2 = \frac{2}{L}\sin^2(5\pi x/L)\cos^2(E_5 t/\hbar)$.

(b) $|\psi(x,t)|^2 = \frac{2}{L}\sin^2(5\pi x/L)\,e^{-i2E_5 t/\hbar}$.

(c) $|\psi(x,t)|^2 = \frac{2}{L}\sin^2(5\pi x/L)$ which is time-independent.

(d) None of the above.

(III) Now suppose that the wave function for an electron at time $t = 0$ is given by $\psi(x, 0) = A \sin^5(\pi x/L)$ where A is a suitable normalization constant. Which one of the following is the wave function at time t?

(a) $\psi(x,t) = A \sin^5(\pi x/L) \cos(E_5 t/\hbar)$

(b) $\psi(x,t) = A \sin^5(\pi x/L)\, e^{-iE_5 t/\hbar}$

(c) Both (a) and (b) above are appropriate ways to write the wave function.

(d) None of the above.

(IV) The wave function for an electron at time $t = 0$ is given by $\psi(x,0) = A \sin^5(\pi x/L)$ where A is a suitable normalization constant. Which one of the following is true about the probability density $|\psi(x,t)|^2$ after time t?

(a) $|\psi(x,t)|^2 = |A|^2 \sin^{10}(\pi x/L) \cos^2(E_5 t/\hbar)$.

(b) $|\psi(x,t)|^2 = |A|^2 \sin^{10}(\pi x/L)\, e^{-i2E_5 t/\hbar}$.

(c) $|\psi(x,t)|^2 = |A|^2 \sin^{10}(\pi x/L)$ which is time-independent.

(d) None of the above.

Open the simulation by double-clicking the green arrow associated with this exercise. Shown is the wave function for an electron in the one-dimensional infinite square well at time $t = 0$ given by $\psi(x,0) = \sqrt{\frac{2}{L}} \sin(5\pi x/L)$ for which in the simulation $\hbar = 2m = L = 1$. Click on the time-development to evolve the wave function in time.

(V) Does $|\psi(x,t)|^2$ for a given x change with time in the simulation?

(a) Yes.

(b) No.

(c) I do not know what I should be looking at in the simulation.

Next, choose the simulation that shows the wave function for an electron in the one-dimensional infinite square well at time $t = 0$ given by $\psi(x, 0) = A \sin^5(\pi x/L)$.

Click on the time-development to evolve the wave function in time.

(VI) Does $|\psi(x, t)|^2$ for a given x change with time in the simulation?

(a) Yes.

(b) No.

(c) I do not know what I should be looking at in the simulation.

Try to reconcile your previous responses with what you observed in the simulation. Remember to begin the simulation by double-clicking the green arrow associated with this exercise.

You watched the time-development simulation for two initial wave functions $\psi(x, 0) = \sqrt{\frac{2}{L}} \sin(5\pi x/L)$ and $\psi(x, 0) = A \sin^5(\pi x/L)$ and noted that $|\psi(x, t)|^2$ for the first initial wave function does not depend on time, whereas it does depend on time for the second wave function.

PART B

Let's think systematically about time evolution of a wave function by working through the following questions.

(1) Choose all of the following statements that are correct about the time evolution of a general wave function:

(I) The time evolution of a general wave function is governed by the Hamiltonian operator for that system.

(II) The time evolution of a general wave function is governed by the time-independent Schrödinger equation (TISE): $\hat{H}\psi_n(x) = E_n\psi_n(x)$.

(III) A general wave function $\psi(x,0)$ at time $t=0$ will evolve into $\psi(x,t) = e^{-iEt/\hbar}\,\psi(x,0)$ at time t.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(2) Choose all of the following statements that are correct about the time evolution of a general wave function:

(I) The time evolution of a wave function is governed by the time-dependent Schrödinger equation (TDSE).

(II) Given $\Psi(x,0)$ and the Hamiltonian, $\hat{H}$, for a system, we can determine the wave function at time t by first calculating the momentum eigenstates.

(III) Given $\Psi(x,0)$ and the Hamiltonian, $\hat{H}$, we can determine the wave function at time t if we first expand $\Psi(x,0)$ in terms of the energy eigenstates.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(3) Choose all of the following statements that are important steps in evaluating the wave function at time t given $\Psi(x,0)$ at time $t=0$:

(I) Solve the TISE to obtain the stationary states $\psi_n(x)$ and the allowed energies E_n.

(II) Solve the TDSE to obtain the stationary states $\psi_n(x)$ and the allowed energies E_n.

(III) Write $\Psi(x,0)$ as a linear superposition of the eigenstates of *any* operator, $\hat{A}$, corresponding to a physical observable: $\Psi(x,0) = \sum c_n \psi_n^{(\hat{A})}(x)$.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (III) only

(e) (II) and (III) only

(4) Choose all of the following statements that are correct:

(I) The stationary states for a Hamiltonian, $\hat{H}$, form a complete set of states.

(II) Any allowed wave function can be written as a linear superposition of stationary states: $\Psi(x) = \sum c_n \psi_n(x)$

(III) c_n in (II) above can be evaluated using orthogonality of wave functions.

Choose all of the above statements that are correct.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) All of the above.

ANSWERS

(1) Correct answer: (a)

Reasoning: (I) is correct because the Hamiltonian operator $\hat{H}$ governs the time evolution of a wave function.

(II) is incorrect because the time evolution of a wave function is governed by the time-dependent Schrödinger equation (TDSE): $i\hbar\,\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t)$.

(III) is incorrect because only for a stationary state (which is a solution of $\hat{H}\psi_n(x) = E_n\psi_n(x)$) does the wave function evolve in time via a simple phase factor given by $\psi_n(x,t) = e^{-iE_nt/\hbar}\,\psi_n(x)$. E_n is the n^{th} allowed energy for the electron. A general state at time $t = 0$ *may not be a stationary state.*

(2) Correct answer: (e)

Reasoning: (I) is correct because the Hamiltonian for a system governs the time evolution via the TDSE.

(II) is incorrect because in order to find $\Psi(x,t)$, we need to know the energy eigenstates (not the momentum eigenstates) because the time evolution of a wave function is governed by the Hamiltonian $\hat{H}$ and not the momentum operator. Only in special situations, e.g., for a free particle, momentum eigenstates are the same as energy eigenstates.

(III) is correct because once $\Psi(x,0)$ is written as a linear superposition of the energy eigenstates (stationary states), the wave function at time t can easily be determined by evolving each stationary state in time via an appropriate phase factor $e^{-iE_nt/\hbar}\,\psi_n(x)$. Note that each stationary state will have a different phase factor if the energy E_n for those states is different. Mathematically, $\Psi(x,t) = e^{-i\hat{H}t/\hbar}\,\Psi(x,0) = e^{-i\hat{H}t/\hbar}\,[\sum c_n\,\psi_n] = \sum c_n\,e^{-iE_nt/\hbar}\,\psi_n$ where the time-evolution operator $e^{-i\hat{H}t/\hbar}$ acting on each energy eigenstate yields the corresponding phase $e^{-iE_nt/\hbar}$. Note that $H\Psi \neq E\Psi$ although $H\psi_n = E_n\psi_n$.

(3) Correct answer: (a)

Reasoning: (I) is correct because solving the TISE, $\hat{H}\psi_n(x) = E_n\psi_n(x)$, gives the stationary states (states of definite energy) and the allowed energies E_n.

(II) is incorrect because stationary states (energy eigenstates) and the allowed energies are obtained by solving TISE.

(III) is incorrect because the time-development is governed by the $\hat{H}$ operator via TDSE. Therefore, the initial state should be expanded in terms of the stationary states (energy eigenstates) and not eigenstates of *any* operator corresponding to a physical observable.

(4) Correct answer: (e)

Reasoning: (I) is correct because stationary states are the eigenstates of the Hamiltonian (energy eigenstates).

(II) is correct because the wave function can be expanded in terms of a complete set of states.

(III) is correct because c_n is evaluated by multiplying both sides of $\Psi(x) = \sum c_n\psi_n(x)$ by $\psi_m^*(x)$, integrating over all space, and using orthogonality of stationary states: $\int \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$. This yields $c_n = \int \psi_n^*(x)\,\Psi(x)\,dx$.

Summary: The following procedure can be used to calculate the wave function $\Psi(x,t)$ at time t given the wave function $\Psi(x,0)$ at time $t=0$ for an electron in a system with a given Hamiltonian $\hat{H}$.

- FIRST: Solve the TISE for that system to find the stationary states $\psi_n(x)$ and allowed energies E_n.
- SECOND: Write the given initial wave function as a linear superposition of the stationary states: $\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$.
- THIRD: To find c_n's for each $\psi_n(x)$, multiply both sides of $\Psi(x,0) = \sum c_n\, \psi_n(x)$ by $\psi_m^*(x)$, integrate over all space, and use orthogonality of stationary states: $\int \psi_m^*(x)\psi_n(x)\, dx = \delta_{mn}$
- FOURTH: Tack on the time dependence with each stationary state component to obtain the time dependence of the general state: $\Psi(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-iE_n t/\hbar}\, \psi_n(x)$.

Flow chart for computing the time evolution of a state

Given $\Psi(x,0)$ and the Hamiltonian $\hat{H}$ Find $\Psi(x,t)$

$\hat{H}\psi_m = E_m\psi_m$ Solve time-indep. Schröd. eq.

$\downarrow$ Time dependence of stationary states

Discrete | Continuous

$\psi_m(x,t) = \exp(-iE_m t/\hbar)\, \psi_m(x)$ | $\psi_k(x,t) = \exp(-iE(k)t/\hbar)\, \psi_k(x)$

$\downarrow$ Expand $\Psi(x,0)$ in terms of stationary states $\downarrow$

$\Psi(x,0) = \sum_n c_n \psi_n(x)$ | $\Psi(x,0) = \int C(k)\, \psi_k(x)\, dk$

$\downarrow$ Find c_m or $c(k)$ using orthogonality $\downarrow$

$c_m = \int \psi_m^*(x)\, \Psi(x,0)\, dx$ | $c(k) = \int \psi_k^*(x)\, \Psi(x,0)\, dx$

$\downarrow$ General solution using superposition $\downarrow$

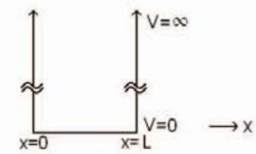
$\Psi(x,t) = \sum_n c_n\, \psi_n(x,t)$
$= \sum_n c_n\, \exp(-iE_n t/\hbar)\, \psi_n(x)$

$\Psi(x,t) = \int c(k)\psi_k(x,t)\, dk$
$= \int c(k)\, \exp(-iE(k)t/\hbar)\, \psi_k(x)\, dk$

PART C

- **Information for the next four questions**

In the following four questions, an electron is in a one-dimensional infinite square well of width L. (The stationary states are $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$, and the allowed energies are $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, where $n = 1, 2, 3...$)



(1.1) The wave function of the electron at time $t = 0$ is given by $\Psi(x, 0) = \sqrt{\frac{2}{7}}\psi_1(x) + \sqrt{\frac{5}{7}}\psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ are the ground state and first excited state. Which one of the following is the wave function $\Psi(x, t)$ at time t?

(a) $e^{-iEt/\hbar}\,\Psi(x, 0)$

(b) $e^{-Et/\hbar}\,\Psi(x, 0)$

(c) $e^{-ixt/\hbar}\,\Psi(x, 0)$

(d) $\sqrt{\frac{2}{7}}\psi_1(x + \omega t) + \sqrt{\frac{5}{7}}\psi_2(x + \omega t)$ where $E = \hbar\omega$

(e) None of the above.

(1.2) Calculate $|\Psi(x, t)|^2$ for the above wave function. Does it depend on time? Explain your reasoning.

Now open the simulation (remember to double-click the green arrow) and choose the initial wave function $\Psi(x, 0) = \sqrt{\frac{2}{7}}\psi_1(x) + \sqrt{\frac{5}{7}}\psi_2(x)$. Watch the time evolution of $|\Psi(x, t)|^2$. Is the time evolution of this wave function consistent with what you predicted earlier? Explain.

(2) An electron is confined in a one-dimensional infinite square well. Choose one of the following statements that is correct about whether $\psi(x) = A\ \sin^2(\pi x/L)$ is an allowed wave function for the electron. (A is a suitable normalization constant):

(a) It is an allowed wave function.

(b) It is not allowed because it does not satisfy $\hat{H}\psi = E\psi$ where $\hat{H}$ is the Hamiltonian operator.

(c) It is not allowed because it is not a linear function but the Schrödinger equation is linear.

(d) It is not allowed because it is not an energy eigenfunction nor is it a linear superposition of energy eigenfunctions.

(e) $A\ \sin^2(\pi x/L)$ is an allowed wave function for two electrons but not for one electron.

(3) The initial wave function for an electron in a one-dimensional infinite square well at $t = 0$ is given by $\Psi(x,0) = A\,\sin^2(\pi x/L)$. Choose all of the following statements related to $\Psi(x,t)$ at time t that are correct.

(I) $\Psi(x,t) = A\,e^{-iE_n t/\hbar}\,\sin^2(\pi x/L)$

(II) $\Psi(x,t) = \sum c_n\,e^{-iE_n t/\hbar}\,\psi_n(x)$ where the expansion coefficients $c_n = \int \psi_n^*(x)\,A\,\sin^2(\pi x/L)\,dx$.

(III) $\Psi(x,t) = \sum c_n\,\sin^2(\pi x/L)\,e^{-iE_n t/\hbar}\,\psi_n(x)$ where the expansion coefficients $c_n = \int \psi_n^*(x)\,A\,\sin^2(\pi x/L)\,dx$.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(4) An electron is confined in a one-dimensional infinite square well. The wave function at time $t = 0$ is denoted by $\Psi(x,0)$. $|\Psi(x_0,t)|^2\,dx$ is the probability of finding the electron in the narrow range between x_0 and $x_0 + dx$ at time t. Choose all of the following statements that are correct:

(I) If $\Psi(x,0)$ is a stationary state, $|\Psi(x_0,t)|^2\,dx$ will not depend on time.

(II) If $\Psi(x,0)$ is localized in space around a position x_0, $|\Psi(x_0,t)|^2\,dx$ will not depend on time.

(III) If $\Psi(x,0)$ is in an energy eigenstate, $|\Psi(x_0,t)|^2\,dx$ will not depend on time.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (III) only

(e) All of the above.

ANSWERS

(1.1) Correct answer: (e)

Reasoning: Using the above flow chart, we should first expand $\Psi(x,0)$ in terms of the stationary states: $\Psi(x,0) = \sum_{n=1}^{\infty} c_n\psi_n(x)$ and determine the coefficients c_n. Fortunately, this $\Psi(x,0)$ is already given in the form $\Psi(x,0) = \sum_{n=1}^{\infty} c_n\psi_n(x) = \sqrt{\frac{2}{7}}\,\psi_1(x) + \sqrt{\frac{5}{7}}\,\psi_2(x)$ and simply by observation, the coefficients $c_1 = \sqrt{\frac{2}{7}}$, $c_2 = \sqrt{\frac{5}{7}}$, and all the other c_n are zero. Since each stationary state $\psi_n(x)$ evolves independently via a phase factor $e^{-iE_nt/\hbar}$, $\Psi(x,t) = \sqrt{\frac{2}{7}}\,e^{-iE_1t/\hbar}\,\psi_1(x) + \sqrt{\frac{5}{7}}\,e^{-iE_2t/\hbar}\,\psi_2(x)$.
Note that the time-dependent phase factors <u>cannot</u> be factored out because they are different for different stationary states.

(2) Correct answer: (a)

Reasoning: (b) is incorrect because the allowed wave functions need not satisfy the TISE $\hat{H}\psi = E\psi$. Rather, any linear superposition of the stationary states (which are solutions of $\hat{H}\psi = E\psi$) is also an allowed state.
(c) $A\,\sin^2(\pi x/L)$ is a smooth function (no discontinuity or cusp) and satisfies the boundary condition for this system (i.e., $\psi(x=0) = 0$ and $\psi(x=L) = 0$ as can be checked by plugging $x = 0$ and $x = L$ in $\psi(x) = A\,\sin^2(\pi x/L)$). Therefore, $A\,\sin^2(\pi x/L)$ can be written as a linear superposition of stationary states.
(d) $A\,\sin^2(\pi x/L)$ can be written as a linear superposition of stationary states (energy eigenstates).
(e) $A\,\sin^2(\pi x/L)$ is an allowed wave function for a single electron. A two-electron wave function must have different coordinates for the two electrons, e.g., $\Psi(x_1,x_2) = A\sin(\pi x_1/L)\sin(\pi x_2/L)$.
<u>Bottom line</u>: If the wave function is single-valued, normalizable, the wave function and its derivative are continuous everywhere, and it satisfies the boundary condition for a particular system, it is an allowed wave function.

(3) Correct answer: (b)

Reasoning: (I) is incorrect because $\Psi(x,0) = A\,\sin^2(\pi x/L)$ is not a stationary state with energy E_n.
(II) is correct because in evaluating $\Psi(x,t)$, $\Psi(x,0) = A\,\sin^2(\pi x/L)$ should only determine the coefficients $c_n = \int \psi_n^*(x)\,A\,\sin^2(\pi x/L)\,dx$.
(III) is incorrect because $\Psi(x,t) = \sum c_n\,e^{-iE_nt/\hbar}\,\psi_n(x)$.

(4) Correct answer: (d) Stationary state and energy eigenstate have the same meaning.

Reasoning: (I) is correct because the stationary states evolve via a simple phase factor: $\Psi(x,t) = e^{-iE_nt/\hbar}\,\psi_n(x)$ and $|\Psi(x_0,t)|^2\,dx = e^{-iE_nt/\hbar}\,\psi_n(x)\,e^{iE_nt/\hbar}\,\psi_n^*(x) = \psi_n(x)\psi_n^*(x)$ will not depend on time because the time-dependent phase factor will drop out due to complex conjugation.
(II) is incorrect because a wave function which is very localized in space around a position x_0 cannot be a stationary state. Therefore, we will have to expand the wave function in terms of the stationary states to find the wave function at time t and different stationary states will evolve via different phase factors: $\Psi(x,t) = \sum_n c_n\,\exp(-iE_nt/\hbar)\,\psi_n(x)$. Taking $|\Psi(x,t)|^2\,dx$ will not get rid of the time-dependent factors (as the cross terms do not necessarily vanish) and $|\Psi(x_0,t)|^2\,dx$ will change with time.
(III) is correct because a state with definite energy is a stationary state (solution of TISE). Because the stationary states evolve via a simple phase factor: $\Psi(x,t) = e^{-iE_nt/\hbar}\,\psi_n(x)$, $|\Psi(x_0,t)|^2\,dx$ will not depend on time (the time-dependent phase factor will drop out due to complex conjugation).

APPENDIX B

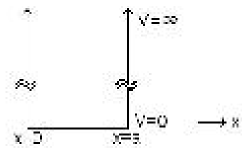
QUANTUM MECHANICS TEST

Preliminary Quiz

(1) Write down the most fundamental equation of quantum mechanics.

In all of the problems below, assume that the measurement of all physical observables is ideal.

(2) The wave function of an electron in a one dimensional infinite square well of width a at time $t = 0$ is given by $\psi(x, 0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary state of the system. ($\phi_n(x) = \sqrt{2/a} sin(n\pi x/a)$, $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1, 2, 3...$)



Answer the following questions about this system:

- (a) Write down the wave function $\Psi(x, t)$ at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

- (b) You measure the energy of an electron at time $t = 0$. Write down the possible values of energy and the probability of measuring each.

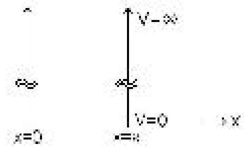
- (c) Calculate the expectation value of energy in the state $\psi(x, t)$ above.

- (d) If the energy measurement yields $4\pi^2\hbar^2/(2ma^2)$, write an expression for the wave function right after the measurement.

- (e) Sketch the above wave function in position space right after the measurement of energy for the previous question.

- (f) Immediately after the energy measurement, you measure the position of the electron. What are the possible values of position you can measure in a narrow range and the probability of measuring each?

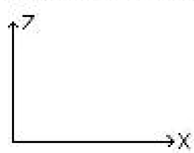
(3) Which of the following wave functions are allowed for an electron in a one dimensional infinite square well of width a: $Asin^3(\pi x/a)$, $A[\sqrt{2/5}sin(\pi x/a)+\sqrt{3/5}sin(2\pi x/a)]$ and $Ae^{-((x-a/2)/a)^2}$? In each of the three cases, A is a suitable normalization constant. You must provide a clear reasoning for each case.



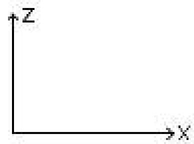
(4) Consider the following statement: "By definition, the Hamiltonian acting on *any* allowed state of the system ψ will give the same state back, i.e., $\hat{H}\psi = E\psi$" where E is the energy of the system. Explain why you agree or disagree with this statement.

(5) Notation: $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ represent the orthonormal eigenstates of $\hat{S}_z$ (the z component of the spin angular momentum) of the electron. SGA is an abbreviation for a Stern-Gerlach apparatus. Ignore the Lorentz force on the electron.

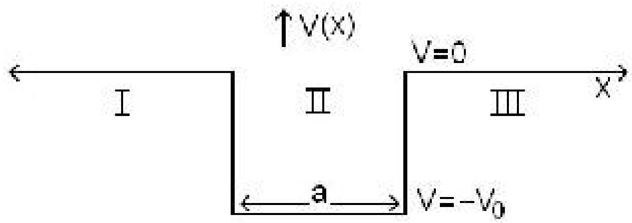
(a) A beam of electrons propagating along the y direction (into the page) in spin state $(|\uparrow_z\rangle+|\downarrow_z\rangle)/\sqrt{2}$ is sent through a SGA with a vertical magnetic field gradient in the $-z$ direction. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x-z plane. Explain your reasoning.



(b) A beam of electrons propagating along the y direction (into the page) in spin state $|\uparrow_z\rangle$ is sent through a SGA with a horizontal magnetic field gradient in the $-x$ direction. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x-z plane. Explain your reasoning.



(6) The potential energy diagram for a finite square well of width a and depth $-V_0$ is shown below.



(a) Below, draw a qualitative sketch of the ground state wave function and comment on the shape of the wave function in all the three regions shown above.

(b) Below, draw a qualitative sketch of any one scattering state wave function (energy $E > 0$) and comment on the shape of the wave function in all the three regions shown above.

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